Algebraic Geometry Lecture 11 – Sheaves

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Motivating Example.

Let X be a topological space (for example $\mathbb{R}$) then look at continuous functions into $\mathbb{R}$, call this set $C(X)$. So

$$C(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}.$$

This object has nice properties:

- $C(X)$ is a ring, with addition and multiplication defined pointwise.
- Given an open subset U of X we can restrict $f \in C(X)$ to $f\big|_U \in C(U)$ in the obvious way: $f\big|_U(u) = f(u)$ for all $u \in U$.
- This restriction is transitive, i.e. if we have open sets $U \subset V \subset X$ then $\left(f\big|_V\right)\Big|_U = f\big|_U$.

We generalise these properties to get:

Def$^{\mathbf{n}}$. A *presheaf* F on a topological space X is a way of assigning a set $F(U)$ to each open set $U \subseteq X$ with maps

$$\mathrm{res}_U^V : F(V) \to F(U)$$

for each pair of open sets $U \subseteq V$, satisfying

- $\mathrm{res}_U^U = \mathrm{id}_{F(U)}$;
- $\mathrm{res}_U^V \circ \mathrm{res}_V^W = \mathrm{res}_U^W$ for all open sets $U \subseteq V \subseteq W$.

We refer to the elements of $F(U)$ as the sections of F over U. Elements of $F(X)$ are called global sections.

If $F(U)$ has some extra structure for all open sets $U \subset X$ we name it accordingly. E.g. C from the above example is a presheaf of rings on X because $C(U)$ is a ring for all open U in X and res_U^V are ring homomorphisms.

We now go back to the motivating example $C(X)$ to look for local structure. Suppose $U \subseteq X$ is open and suppose $\{U_i\}_{i \in I}$ is an open cover of U, i.e. the U_i are open for all $i \in I$ and $U \subseteq \bigcup_{i \in I} U_i$. If we know how some $f \in C(U)$ acts on each U_i (i.e. locally) then we can know how it acts on U (i.e. globally). This is not trivial, it relies on the fact that

$$f\big|_{U_i}(x) = f\big|_{U_j}(x)$$

whenever $x \in U_i \cap U_j$. We also have the converse: any collection of functions $\{f_i\}$ defined on an open cover $\{U_j\}$ of U whose elements agree on intersections of U_i

[1]Notes typed by Lee Butler based on a lecture given by Joe Grant. Any errors are the responsibility of the typist. Or Hartshorne.

and U_j determines a unique function $f \in C(U)$. This is the local-global interplay that we wish to capture.

Def$^{\mathbf{n}}$. A presheaf F on a topological space X is a *sheaf* if and only if it satisfies the "gluing axiom": given an open cover $\{U_i\}_{i\in I}$ of an open set $U \subseteq X$, whenever we have elements $s_i \in F(U_i)$, if

$$\mathrm{res}^{U_i}_{U_i\cap U_j} s_i = \mathrm{res}^{U_j}_{U_i\cap U_j} s_j \quad \text{for all } i, j \in I,$$

then there exists a unique section $s \in F(U)$ such that $\mathrm{res}^{U}_{U_i} s = s_i$ for every $i \in I$.

Why is this useful?

Remark: We'll use X for varieties and V for open sets.

Proposition. *Let X be a variety over a field $k = \overline{k}$. For each open set $U \in X$ in the Zariski topology let $\mathcal{O}(U)$ be the ring of regular functions from U to k, and for each $V \subseteq U$ let res^U_V be the usual restriction of functions. Then $\mathcal{O}$ is a sheaf of rings on X, called the sheaf of regular functions on X.*

Proof. Recall that $f : Y \to k$ is regular at $P \in Y$ if there is an open neighbourhood V containing P with $V \subseteq Y$ and there are polynomials $g, h \in k[x_1, \dots, x_n]$ such that $h \neq 0$ on V and $f = g/h$ on V. f is regular if f is regular for all $p \in Y$. So clearly $\mathcal{O}(U)$ is a ring and as our restriction map is the usual one, $\mathcal{O}$ is a presheaf of rings on X.

The gluing axiom follows from the definition: if f is regular locally (i.e. for all $P \in U_i$) then f is regular globally (i.e. for all $P \in U$). The existence part is easy. $\square$

Now let's see some more sheaves!

Consider $\underline{A}$ which sends each open set $U \subset X$ to the same abelian group A. This is a presheaf buy it's not a sheaf (exercise). But we can sheafify it!

What is the sheafification of $\underline{A}$? If U is the union of n connected open sets then the sheafification of $\underline{A}$, denoted $\underline{A}^+$, is the direct product of n copies of A.

More examples of sheaves:

- Sheaf of differentiable functions as a differentiable manifold.
- Sheaf of solutions to a linear differential equation.

We want to bring local rings into this.

Def$^{\mathbf{n}}$. If f is a presheaf on X and P is a point of X we define the stalk of F at P, F_P, by

$$F_P = \{(U, s) \mid U \text{ is an open neighbourhood of } P, s \in F(U)\} \Big/ \sim$$

where $(U, s) \sim (V, t)$ if and only if there is an open neighbourhood W of P with $W \subseteq U \cap V$ and such that $\mathrm{res}^U_W s = \mathrm{res}^V_W t$. We call the elements of the stalk at P the germs of sections of F at P.

Recall the definition of a local ring, then if X is a variety and $\mathcal{O}$ is its sheaf of regular functions then the stalk $\mathcal{O}_P$ at a point P is just the local ring of P on X.